

Week 7 Worksheet

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

Main topic: Euler's method, Application of differential equations (Problem 1-3 from past exams)

1. Estimate $y(1)$ by Euler's method with step size $\frac{1}{3}$ for

$$\Delta x = \frac{1}{3}$$

$$\frac{dy}{dx} = y - x, \quad y(0) = 1.$$

x	y	$\frac{\Delta y}{\Delta x}$	$y_{\text{new}} = y + \frac{\Delta y}{\Delta x} \cdot \Delta x$
0	1	1	$1 + 1 \cdot \frac{1}{3} = \frac{4}{3}$
$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3} - \frac{1}{3} = 1$	$\frac{4}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$
$\frac{2}{3}$	$\frac{5}{3}$	$\frac{5}{3} - \frac{2}{3} = 1$	$\frac{5}{3} + 1 \cdot \frac{1}{3} = 2$
1	2		

$$y(1) \approx 2$$

2. Use Euler's method with step size $h = 0.1$ to estimate $y(1.2)$ for problem

$$\Delta x = 0.1$$

$$\frac{dy}{dx} = -2y + 3x \text{ and } y(1) = 0.$$

x	y	$\frac{\Delta y}{\Delta x}$	$y_{\text{new}} = y + \frac{\Delta y}{\Delta x} \cdot \Delta x$
1	0	3	$0 + 3 \cdot 0.1 = 0.3$
1.1	0.3	$-0.6 + 3.3 = 2.7$	$0.3 + 2.7 \cdot 0.1 = 0.57$
1.2	0.57		

$$y(1.2) \approx 0.57$$

3. Let t stand for time in minutes from 12:00pm and let $N(t)$ denote the number of gallons of salt in a vat at time t . Assume that N satisfies $\frac{dN}{dt} = 50(1 - N^2)$. Also assume that at 12:00pm there were 3 gallons of salt in the vat. Compute $N(t)$.

Plug in $(t, N) = (0, 3)$

$$N(0) = 3$$

$$\frac{dN}{1-N^2} = 50 dt \Rightarrow \int \frac{-\frac{1}{2}}{N-1} + \frac{\frac{1}{2}}{N+1} dN = \int 50 dt$$

$$\frac{-1}{(N-1)(N+1)} = \frac{1}{N^2} = \frac{A}{N-1} + \frac{B}{N+1}$$

$$-1 = A(N+1) + B(N-1)$$

$$N=1 \Rightarrow -1 = 2A \quad A = -\frac{1}{2}$$

$$N=-1 \Rightarrow -1 = -2B \quad B = \frac{1}{2}$$

$$-\frac{1}{2} \ln|N-1| + \frac{1}{2} \ln|N+1| = 50t + C$$

$$-\ln|N-1| + \ln|N+1| = 100t + 2C$$

$$\ln \left| \frac{N+1}{N-1} \right| = 100t + 2C$$

$$\frac{N+1}{N-1} = \underbrace{\pm e^{2C}}_{A} \cdot e^{100t}$$

$$\frac{1}{2} = Ae^0 \Rightarrow A = 2$$

$$\frac{N+1}{N-1} = 2e^{100t}$$

$$N+1 = 2e^{100t} \quad N = 2e^{100t} - 1$$

$$N = \frac{1 + 2e^{100t}}{2e^{100t} - 1}$$

$$\frac{dT_S}{dt} = -K(T_S - T_r)$$

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$$(b) \frac{\frac{dT_S}{dt}}{T_S - T_r} = -k dt$$

$$\ln(T_S - T_r) = -kt + C$$

$$|T_S - T_r| = e^{-kt} \cdot e^C$$

$$T_S - T_r = \pm e^C \cdot e^{-kt} \quad \text{denote as } Ae^{-kt}$$

$$\Rightarrow T_S = T_r + Ae^{-kt}$$

As $t \rightarrow \infty$, $T_S(t) \rightarrow T_r$.

5. (a) Derive a reduction formula for $I_n = \int \cos^n x dx$

$$(b) \text{ Use it to evaluate } \int_0^{\frac{\pi}{4}} \cos^4 x dx$$

$$(a) I_n = \int \underbrace{\cos^n x}_{F} \underbrace{\cos x}_{G'} dx \quad F = \cos^{n-1} x, G' = \cos x$$

$$F' = (n-1) \cos^{n-2} x \cdot (-\sin x), G = \sin x$$

$$= \cos^{n-1} x \cdot \sin x + \int (n-1) \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$I_n = \cos^{n-1} x \cdot \sin x + (n-1) [I_{n-2} - I_n]$$

$$I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$n I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} \Rightarrow I_n = \frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} I_{n-2}$$

$$6. \int \frac{x^2+1}{x^2-1} dx$$

$$= \int 1 + \frac{2}{x^2-1} dx$$

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x-1)$$

$$\text{Plug in } x=1, A=1$$

$$x=-1, B=-1$$

$$? = \int 1 + \frac{1}{x-1} - \frac{1}{x+1} dx$$

$$= x + \ln|x-1| - \ln|x+1| + C$$

$$(c) \begin{cases} T_S(0) = 180 \\ T_r = 75 \end{cases}$$

Find k ?

$$T_S(5) = 150$$

Find the time when $T_S = 90$.

$$T_S = 75 + Ae^{-kt}$$

$$180 = 75 + A \Rightarrow A = 105$$

$$150 = 75 + 105e^{-kt} \Rightarrow \frac{75}{105} = e^{-kt}$$

$$-\frac{1}{5} \ln \frac{75}{105} = k$$

$$90 = 75 + 105 e^{(-\frac{1}{5} \ln \frac{75}{105})t}$$

$$15 = 105 e^{(\frac{1}{5} \ln \frac{75}{105})t}$$

$$\ln \left(\frac{15}{105} \right) = \left(\frac{1}{5} \ln \frac{75}{105} \right) t$$

$$t = \frac{5 \ln \left(\frac{15}{105} \right)}{\ln \left(\frac{75}{105} \right)}$$

$$\text{Denote } A_n = \int_0^{\frac{\pi}{4}} \cos^n x dx$$

$$A_n = \frac{1}{n} \cos^{n-1} x \sin x \Big|_{x=0}^{x=\frac{\pi}{4}} + \frac{n-1}{n} A_{n-2}$$

$$A_n = \frac{1}{n} \left(\frac{\sqrt{2}}{2} \right)^n + \frac{n-1}{n} A_{n-2}$$

$$A_0 = \int_0^{\frac{\pi}{4}} 1 dx = \frac{\pi}{4}$$

$$A_2 = \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right)^2 + \frac{1}{2} A_0 = \frac{1}{4} + \frac{\pi}{8}$$

$$A_4 = \frac{1}{4} \left(\frac{\sqrt{2}}{2} \right)^4 + \frac{3}{4} A_2$$

$$= \frac{1}{16} + \frac{3}{4} \left(\frac{1}{4} + \frac{\pi}{8} \right)$$